$$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{\cos x}{1-\sin^2 x}\,dx$$

ここで $t = \sin x$ とおくと

$\frac{dt}{dx} = \cos x$ より $dt = \cos x\,dx$

x	$0 \cdots \frac{\pi}{4}$
t	$0 \cdots \frac{1}{\sqrt{2}}$

よって $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{1-t^2}$

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt$$

$$= \frac{1}{2}\Big[-\log|1-t| + \log|1+t|\Big]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\left[\log\left|\frac{1+t}{1-t}\right|\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\left\{\log\left|\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right| - \log 1\right\}$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{2}\log(\sqrt{2}+1)^2$$

$$= \boxed{\log(\sqrt{2}+1)}$$

別解 $\cos^2 x = 1-\sin^2 x$ より

$\cos^2 x = (1+\sin x)(1-\sin x)$

$0 \leqq x \leqq \frac{\pi}{4}$ で $\cos x \neq 0$, $1+\sin x \neq 0$ であることから

$$\frac{\cos x}{1+\sin x} = \frac{1-\sin x}{\cos x} \text{ より}$$

$$\frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

よって $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \left(\frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x}\right) dx$

$$= \Big[\log|1+\sin x| - \log|\cos x|\Big]_0^{\frac{\pi}{4}}$$

$$= \left[\log\left|\frac{1+\sin x}{\cos x}\right|\right]_0^{\frac{\pi}{4}}$$

$$= \log\left|\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right| - \log 1$$

$$= \boxed{\log(\sqrt{2}+1)}$$