(1) $\displaystyle\int_{x_k}^{x_{k+1}} \cos^2 nx\,dx = \int_{x_k}^{x_{k+1}} \frac{1+\cos 2nx}{2}\,dx$

$\displaystyle = \left[\frac{1}{2}x + \frac{1}{2}\times\frac{1}{2n}\sin 2nx\right]_{x_k}^{x_{k+1}}$

$\displaystyle = \frac{1}{2}\left(x_{k+1}-x_k\right) + \frac{1}{4n}\left(\sin 2nx_{k+1} - \sin 2nx_k\right)$

$\displaystyle = \frac{1}{2}\left(\frac{\pi(k+1)}{2n} - \frac{\pi k}{2n}\right) + \frac{1}{4n}\left(\sin(k+1)\pi - \sin k\pi\right)$

$\displaystyle = \boxed{\frac{\pi}{4n}}$

(2) $f(x)$ は単調増加であるから

$\displaystyle\frac{\pi k}{2n} \leqq x \leqq \frac{\pi(k+1)}{2n}$ つまり $x_k \leqq x \leqq x_{k+1}$ で

$f(x_k) \leqq f(x) \leqq f(x_{k+1})$ がなりたつ.

すべての辺に $\cos^2 x\ (\geqq 0)$ をかけて

$f(x_k)\cos^2 x \leqq f(x)\cos^2 x \leqq f(x_{k+1})\cos^2 x$ がなりたつ

よって $\displaystyle\int_{x_k}^{x_{k+1}} f(x_k)\cos^2 x\,dx \leqq \int_{x_k}^{x_{k+1}} f(x)\cos^2 x\,dx \leqq \int_{x_k}^{x_{k+1}} f(x_{k+1})\cos^2 x\,dx$

もなりたち

$f(x_k)$, $f(x_k)$ は定数であることと $\displaystyle\int_{x_k}^{x_{k+1}} \cos^2 x\,dx = \frac{\pi}{4n}$ であることから

$\displaystyle\frac{\pi}{4n}f(x_k) \leqq S_k \leqq \frac{\pi}{4n}f(x_{k+1})$ はなりたつ

(3) (2)より $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{4n}f(x_k) \leqq \lim_{n\to\infty}\sum_{k=0}^{n-1} S_k \leqq \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{4n}f(x_{k+1})$ がなりたつ. — ①

ここで $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{4n}f(x_k) = \lim_{n\to\infty}\frac{\pi}{4}\times\frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k\pi}{2n}\right)$

$\displaystyle = \frac{\pi}{4}\int_0^1 f\left(\frac{\pi}{2}x\right)dx$ であり

$u = \frac{\pi}{2}x$ とおくと $du = \frac{\pi}{2}dx$,

x	$0 \cdots 1$
0	$0 \cdots \frac{\pi}{2}$

だから

$\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{4n}f(x_k) = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} f(u)\cdot\frac{2}{\pi}du$

$\displaystyle = \frac{1}{2}\int_0^{\frac{\pi}{2}} f(u)\,du$

$\displaystyle = \frac{1}{2}I$ — ②

また $\displaystyle \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{4n}f(x_{k+1}) = \lim_{n\to\infty}\frac{\pi}{4}\times\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{(k+1)\pi}{2n}\right)$

$$= \lim_{n\to\infty}\frac{\pi}{4}\times\frac{1}{n}\sum_{l=1}^{n}f\left(\frac{l\pi}{2n}\right)$$

$$= \frac{\pi}{4}\int_0^1 f\left(\frac{\pi}{2}x\right)dx$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} f(u)\cdot\frac{2}{\pi}du$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} f(u)du = \frac{1}{2}I \quad \text{—③ である}$$

①②③より はさみうちの定理から

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}S_k = \lim_{n\to\infty}\left\{\int_0^{x_1}f(x)\cos^2 nx\,dx + \int_{x_1}^{x_2}f(x)\cos^2 nx\,dx + \cdots + \int_{x_{n-1}}^{x_n}f(x)\cos^2 nx\,dx\right\}$$

$$= \lim_{n\to\infty}\int_0^{x_n}f(x)\cos^2 nx\,dx$$

$$= \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}f(x)\cos^2 nx\,dx = \boxed{\frac{I}{2}} \quad \text{となる}$$

(4) $\cos^2 nx - \cos^4 nx = \cos^2 nx(1-\cos^2 nx)$

$$= \cos^2 nx\sin^2 nx$$

$$= \left(\frac{1}{2}\sin 2nx\right)^2 = \frac{1}{4}\sin^2 2nx \quad \text{であるから}$$

$$J = \lim_{n\to\infty}\int_0^{\frac{\pi}{4}}(\cos^2 nx - \cos^4 nx)\log\left(1+\frac{4}{\pi}x\right)dx \quad \text{とおくと}$$

$$J = \lim_{n\to\infty}\int_0^{\frac{\pi}{4}}\frac{1}{4}\sin^2 2nx\log\left(1+\frac{4}{\pi}x\right)dx \quad \text{であり}$$

$2x = u$ とおくと $du = 2dx$,

x	$0 \cdots \frac{\pi}{4}$
u	$0 \cdots \frac{\pi}{2}$

だから

$$J = \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{1}{4}\sin^2 nu\log\left(1+\frac{2}{\pi}u\right)\cdot\frac{1}{2}du$$

$$= \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{1}{8}(1-\cos^2 nu)\log\left(1+\frac{2}{\pi}u\right)du$$

$$= \frac{1}{8}\lim_{n\to\infty}\left\{\int_0^{\frac{\pi}{2}}\log\left(1+\frac{2}{\pi}u\right)du - \int_0^{\frac{\pi}{2}}\cos^2 nu\log\left(1+\frac{2}{\pi}u\right)du\right\} \quad \text{となる}$$

ここで $f(u)=\log\left(1+\frac{2}{\pi}u\right)$ とおくと $f(u)$ は単調に増加する連続関数

であるから (3) がなりたつので

$$J=\frac{1}{8}\left(I-\frac{I}{2}\right)=\frac{1}{16}I$$ がなりたつ.

ここで $I=\int_0^{\frac{\pi}{2}}\log\left(1+\frac{2}{\pi}u\right)du$

$$=\int_0^{\frac{\pi}{2}}\frac{\pi}{2}\times\left(1+\frac{2}{\pi}u\right)'\log\left(1+\frac{2}{\pi}u\right)du$$

$$=\left[\frac{\pi}{2}\left(1+\frac{2}{\pi}u\right)\log\left(1+\frac{2}{\pi}u\right)\right]_0^{\frac{\pi}{2}}-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1+\frac{2}{\pi}u}{1+\frac{2}{\pi}u}\times\frac{2}{\pi}du$$

$$=\frac{\pi}{2}\left(1+\frac{2}{\pi}\times\frac{\pi}{2}\right)\log\left(1+\frac{2}{\pi}\times\frac{\pi}{2}\right)-\frac{\pi}{2}\times\frac{2}{\pi}\int_0^{\frac{\pi}{2}}du$$

$$=\frac{\pi}{2}\times 2\log 2-\left[u\right]_0^{\frac{\pi}{2}}$$

$$=\pi\log 2-\frac{\pi}{2}$$

$$=\frac{2\log 2-1}{2}\pi$$ となるから

求める値 J は $J=\frac{1}{16}I=\boxed{\frac{2\log 2-1}{32}\pi}$ となる