$$\int_1^e x\ (\log x)^2\,dx$$

$$= \int_1^e \left(\frac{x^2}{2}\right)' (\log x)^2\,dx$$

$$= \left[\frac{x^2}{2}(\log x)^2\right]_1^e - \int_1^e \left(\frac{x^2}{2}\right) \times 2(\log x) \times \frac{1}{x}\,dx$$

$$= \frac{e^2}{2}\underbrace{(\log e)^2}_{=1} - \frac{1^2}{2}\underbrace{(\log 1)^2}_{=0} - \int_1^e x\log x\,dx$$

$$= \frac{e^2}{2} - \int_1^e \left(\frac{x^2}{2}\right)' \log x\,dx$$

$$= \frac{e^2}{2} - \left(\left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{x^2}{2} \times \frac{1}{x}\,dx \right)$$

$$= \frac{e^2}{2} - \left(\frac{e^2}{2}\log e - \frac{1^2}{2} \times \log 1 - \frac{1}{2}\int_1^e x\,dx \right)$$

$$= \frac{1}{2}\int_1^e x\,dx = \left[\frac{x^2}{4}\right]_1^e = \boxed{\frac{e^2-1}{4}}$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{\sin^2 x + 3\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\frac{\sin^2 x}{\cos^2 x} + 3 \times \frac{\cos^2 x}{\cos^2 x}} \times \frac{1}{\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2 x + 3} \times \frac{1}{\cos^2 x}\,dx \quad — ①$$

ここで $t = \tan x$ とおくと

$\frac{dt}{dx} = \frac{1}{\cos^2 x}$ より $dt = \frac{1}{\cos^2 x}dx$,

x	$0 \cdots \frac{\pi}{4}$
t	$0 \cdots 1$

だから

$$(①式) = \int_0^1 \frac{1}{t^2+3}\,dt \quad となる \quad — ②$$

またここで $t = \sqrt{3}\tan\theta$ とおくと

$\frac{dt}{d\theta} = \sqrt{3} \times \frac{1}{\cos^2\theta}$ より $dt = \frac{\sqrt{3}}{\cos^2\theta}d\theta$,

t	$0 \cdots 1$
θ	$0 \cdots \frac{\pi}{6}$

だから

$$(②式) = \int_0^{\frac{\pi}{6}} \frac{1}{3\tan^2\theta + 3} \cdot \frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3(\tan^2\theta + 1)} \cdot \frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}}{3}\,d\theta = \left[\frac{\sqrt{3}}{3}\theta\right]_0^{\frac{\pi}{6}} = \boxed{\frac{\sqrt{3}}{18}\pi}$$