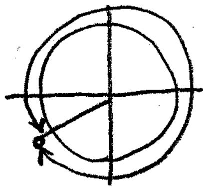
(1) $\sqrt{3}\cos x - \sin x = 2\cos\left(x+\frac{\pi}{6}\right)$ であるから $-\pi \leqq x \leqq \pi$ のとき $-\frac{5}{6}\pi \leqq x+\frac{\pi}{6} \leqq \frac{7}{6}\pi$ より



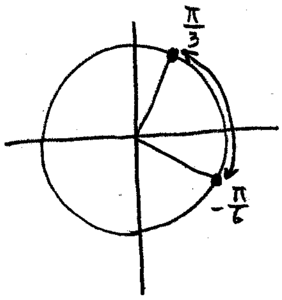
$2\cos\left(x+\frac{\pi}{6}\right) > 0$ の範囲は

$-\frac{\pi}{2} < x+\frac{\pi}{6} < \frac{\pi}{2}$ である

よって求めるxの範囲は $\boxed{-\frac{2}{3}\pi < x < \frac{\pi}{3}}$ となる

(2) $\sqrt{3}\cos x - \sin x = 2\cos\left(x+\frac{\pi}{6}\right)$ は

$-\frac{\pi}{3} \leqq x \leqq \frac{\pi}{6}$ のとき $-\frac{\pi}{6} \leqq x+\frac{\pi}{6} \leqq \frac{\pi}{3}$ であるから (1)より



$2\cos\left(x+\frac{\pi}{6}\right) > 0$ である.

よって $\sin x$のみの符号を考えればよいので

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \left| \frac{4\sin x}{\sqrt{3}\cos x - \sin x} \right| dx$$

$$= \int_{-\frac{\pi}{3}}^{0} \frac{-4\sin x}{2\cos\left(x+\frac{\pi}{6}\right)} dx + \int_{0}^{\frac{\pi}{6}} \frac{4\sin x}{2\cos\left(x+\frac{\pi}{6}\right)} dx$$

$$= \int_{0}^{-\frac{\pi}{3}} \frac{2\sin x}{\cos\left(x+\frac{\pi}{6}\right)} dx + \int_{0}^{\frac{\pi}{6}} \frac{2\sin x}{\cos\left(x+\frac{\pi}{6}\right)} dx \quad \text{—①} \quad \text{となる.}$$

ここで $I = \int \frac{2\sin x}{\cos\left(x+\frac{\pi}{6}\right)} dx$ とおき, $x+\frac{\pi}{6} = t$ とすると $dt = dx$ より

$$I = \int \frac{2\sin\left(t-\frac{\pi}{6}\right)}{\cos t} dx = \int \frac{2\sin t\cos\frac{\pi}{6} - 2\cos t\sin\frac{\pi}{6}}{\cos t} dt$$

$$= \int \frac{\sqrt{3}\sin t - \cos t}{\cos t} dt$$

$$= \int \left\{ \sqrt{3} \times \frac{-(\cos t)'}{\cos t} - 1 \right\} dt$$

$$= -\sqrt{3}\log|\cos t| - t + C \quad (C\text{は定数})$$

となるから

$$\begin{array}{c|ccc} x & 0 & \cdots & -\frac{\pi}{3} \\ \hline t & \frac{\pi}{6} & \cdots & -\frac{\pi}{6} \end{array} , \quad \begin{array}{c|ccc} x & 0 & \cdots & \frac{\pi}{6} \\ \hline t & \frac{\pi}{6} & \cdots & \frac{\pi}{3} \end{array}$$ より ①式は

$$(\text{①式}) = \left[-\sqrt{3}\log|\cos t| - t \right]_{\frac{\pi}{6}}^{-\frac{\pi}{6}} + \left[-\sqrt{3}\log|\cos t| - t \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \cancel{-\sqrt{3}\log\frac{\sqrt{3}}{2}} + \cancel{\sqrt{3}\log\frac{\sqrt{3}}{2}} + \frac{\pi}{6} + \frac{\pi}{6} - \sqrt{3}\log\frac{1}{2} + \sqrt{3}\log\frac{\sqrt{3}}{2} - \frac{\pi}{3} + \frac{\pi}{6}$$

$$= \sqrt{3}\log\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} + \frac{\pi}{6} = \sqrt{3}\log\sqrt{3} + \frac{\pi}{6} = \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{2}\log 3}$$