(1) $I_0=\int_0^{\frac{\pi}{4}} dx = [x]_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{4}}$

$I_{-1}=\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x)^{-1}}dx = \int_0^{\frac{\pi}{4}} \cos x\, dx = [\sin x]_0^{\frac{\pi}{4}} = \sin\frac{\pi}{4} = \boxed{\frac{\sqrt{2}}{2}}$

$I_2=\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = [\tan x]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} = \boxed{1}$

(2) $I_1=\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^2 x}dx$

$= \int_0^{\frac{\pi}{4}} \frac{\cos x}{1-\sin^2 x}dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)dx$

$= \frac{1}{2}\left[-\log|1-\sin x| + \log|1+\sin x|\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\log\left|\frac{1+\sin x}{1-\sin x}\right|\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left\{\log\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| - \log\frac{1}{1}\right\}$

$= \frac{1}{2}\cdot\log\frac{2+\sqrt{2}}{2-\sqrt{2}}$

$= \frac{1}{2}\log\frac{(2+\sqrt{2})^2}{2}$

$= \frac{1}{2}\left\{2\log(2+\sqrt{2}) - \log 2\right\}$

$= \log(2+\sqrt{2}) - \log\sqrt{2} = \log\frac{2+\sqrt{2}}{\sqrt{2}} = \log\frac{2\sqrt{2}+2}{2} = \boxed{\log(1+\sqrt{2})}$

(3) $I_n=\int_0^{\frac{\pi}{4}}\frac{dx}{(\cos x)^n}=\int_0^{\frac{\pi}{4}}(\sin x)'\frac{dx}{(\cos x)^{n+1}}$

$$=\left[\frac{\sin x}{(\cos x)^{n+1}}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\sin x\cdot\frac{-(n+1)}{(\cos x)^{n+2}}\cdot(-\sin x)\,dx$$

$$=\frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^{n+1}}-(n+1)\int_0^{\frac{\pi}{4}}\frac{\sin^2 x}{(\cos x)^{n+2}}\cdot dx$$

$$=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^{n}}-(n+1)\int_0^{\frac{\pi}{4}}\frac{1-\cos^2 x}{(\cos x)^{n+2}}\cdot dx$$

$$=(\sqrt{2})^n-(n+1)\int_0^{\frac{\pi}{4}}\frac{1}{(\cos x)^{n+2}}\cdot dx+(n+1)\int_0^{\frac{\pi}{4}}\frac{1}{(\cos x)^{n}}\cdot dx$$ となるから、

$I_n=(\sqrt{2})^n-(n+1)I_{n+2}+(n+1)I_n$ が成りたつ。

ゆえに、全ての整数 n で、$nI_n-(n+1)I_{n+2}+(\sqrt{2})^n=0$ ……① が成りたつ。

(4) ①式に $n=-3$ を代入しても成り立つから、$-3I_{-3}-(-2)I_{-1}+(\sqrt{2})^{-3}=0$ より、

$I_{-3}=\frac{1}{3}\left(2I_{-1}+\frac{1}{2\sqrt{2}}\right)=\frac{1}{3}\left(2\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{4}\right)=\frac{5\sqrt{2}}{12}$ となる。

また、①式に $n=-2$ を代入して、$-2I_{-2}-(-1)I_0+(\sqrt{2})^{-2}=0$ より、

$I_{-2}=\frac{1}{2}\left(I_0+\frac{1}{2}\right)=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\pi+2}{8}$ となる。

さらに、①式に $n=1$ を代入して、$I_1-2I_3+(\sqrt{2})^1=0$ より、

$I_3=\frac{1}{2}(I_1+\sqrt{2})=\frac{1}{2}\{\sqrt{2}+\log(1+\sqrt{2})\}$ となる。

(5) $x=\tan\theta$ とおくと、$dx=\frac{d\theta}{\cos^2\theta}$ であり、

x	0	$\cdots$	1
θ	0	$\cdots$	$\frac{\pi}{4}$

であるから、

$$\int_0^1\sqrt{x^2+1}\,dx=\int_0^{\frac{\pi}{4}}\sqrt{\tan^2\theta+1}\,\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\sqrt{\frac{1}{\cos^2\theta}}\,\frac{d\theta}{\cos^2\theta}$$

$$=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos^3\theta}=I_{-3}=\frac{1}{2}\{\sqrt{2}+\log(1+\sqrt{2})\}$$

また、$\int_0^1\frac{dx}{(x^2+1)^2}=\int_0^{\frac{\pi}{4}}\frac{1}{(\tan^2\theta+1)^2}\cdot\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\cos^4\theta\cdot\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos^{-2}\theta}$

$$=I_{-2}=\frac{\pi+2}{8}$$ となる。