(1) $(1+2\cos 2x+2\cos 4x+\cdots\cdots+2\cos 2nx)\sin nx=\sin(2n+1)x$ ……①の証明

一般に $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$ が成り立つから、

$$
\begin{aligned}
(\text{①の左辺}) &= \sin x+\sum_{k=1}^{n}2\cos 2kx\sin x \\
&= \sin x+\sum_{k=1}^{n}\{\sin(2k+1)x-\sin(2k-1)x\} \\
&= \sin x+(\cancel{\sin 3x}-\sin x) \\
&\qquad +(\cancel{\sin 5x}-\cancel{\sin 3x}) \\
&\qquad +\cdots\cdots \\
&\qquad +\{\sin(2n+1)x-\cancel{\sin(2n-1)x}\} \\
&= \sin x+\{-\sin x+\sin(2n+1)x\} \\
&= \sin(2n+1)x=(\text{①の右辺})
\end{aligned}
$$

よって①は成り立つ。

$\{\sin x+\sin 3x+\sin 5x+\cdots\cdots+\sin(2n-1)x\}\sin x=\sin^2 nx$ ……②の証明

一般に $\sin\alpha\sin\beta=-\dfrac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$ が成り立つから、

$$
\begin{aligned}
(\text{②の左辺}) &= \sum_{k=1}^{n}\sin\{(2k-1)x\}\sin x \\
&= -\frac{1}{2}\sum_{k=1}^{n}\{\cos 2kx-\cos(2k-2)x\} \\
&= -\frac{1}{2}[(\cancel{\cos 2x}-\cos 0) \\
&\qquad +(\cancel{\cos 4x}-\cancel{\cos 2x}) \\
&\qquad +\cdots\cdots \\
&\qquad +\{\cos 2nx-\cancel{\cos(2n-2)x}\}] \\
&= -\frac{1}{2}(-1+\cos 2nx) \\
&= \frac{1-\cos 2nx}{2}=\sin^2 nx=(\text{②の右辺})
\end{aligned}
$$

よって②は成り立つ。

(2) (1) より $\sin(2n+1)x=\sin x+\displaystyle\sum_{k=1}^{n}2\cos 2kx\sin x$ だから

$$
\begin{aligned}
(\text{与式}) &= \lim_{a\to+0}\int_a^{\frac{\pi}{2}}(1+2\sum_{k=1}^{n}\cos 2kx)\,dx \\
&= \lim_{a\to+0}\left[x+\sum_{k=1}^{n}\frac{1}{k}\sin 2kx\right]_a^{\frac{\pi}{2}}
\end{aligned}
$$

$$=\lim_{a\to+0}\left(\frac{\pi}{2}-a+\sum_{k=1}^{n}\frac{1}{k}\sin k\pi-\sum_{k=1}^{n}\frac{1}{k}\sin 2ka\right)$$

$$=\frac{\pi}{2}-0+0-0=\frac{\pi}{2}$$

(3) (1) より $\sin^2 nx=\sum_{k=1}^{n}\sin\{(2k-1)x\}\sin x$ が成り立つから

$$(\text{与式})=\lim_{a\to+0}\int_a^{\frac{\pi}{2}}\sum_{k=1}^{n}\frac{\sin\{(2k-1)x\}}{\sin x}dx$$

$$=\lim_{a\to+0}\int_a^{\frac{\pi}{2}}\left(\frac{\sin x}{\sin x}+\frac{\sin 3x}{\sin x}+\frac{\sin 5x}{\sin x}+\cdots\cdots+\frac{\sin(2n-1)x}{\sin x}\right)dx$$

ここで (2) より、n が自然数のとき $\lim_{a\to+0}\int_a^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{\sin x}dx=\frac{\pi}{2}$ が成り立つから、

$$(\text{与式})=\lim_{a\to+0}\int_a^{\frac{\pi}{2}}\left(\frac{\sin x}{\sin x}\right)dx+\frac{\pi}{2}\cdot(n-1)$$

$$=\lim_{a\to+0}\left(\frac{\pi}{2}-a\right)+\frac{\pi}{2}\cdot(n-1)=\frac{n\pi}{2}$$