(1) $$\int_\alpha^\beta (x-\alpha)(x-\beta)\,dx = \int_\alpha^\beta (x-\alpha)\{(x-\alpha)+(\alpha-\beta)\}\,dx$$

$$= \int_\alpha^\beta \{(x-\alpha)^2+(\alpha-\beta)(x-\alpha)\}\,dx$$

$$= \left[\frac{1}{3}(x-\alpha)^3+(\alpha-\beta)\times\frac{1}{2}(x-\alpha)^2\right]_\alpha^\beta$$

$$= \frac{1}{3}(\beta-\alpha)^3-(\beta-\alpha)\times\frac{1}{2}(\beta-\alpha)^2$$

$$= \frac{1}{3}(\beta-\alpha)^3-\frac{1}{2}(\beta-\alpha)^3 = \boxed{-\frac{1}{6}(\beta-\alpha)^3}$$

(2) $$\int_\alpha^\beta (x-\alpha)^n(x-\beta)\,dx = \int_\alpha^\beta \{(x-\alpha)^n\times\{(x-\alpha)+(\alpha-\beta)\}\,dx$$

$$= \int_\alpha^\beta \{(x-\alpha)^{n+1}-(\beta-\alpha)(x-\alpha)^n\}\,dx$$

$$= \left[\frac{1}{n+2}(x-\alpha)^{n+2}-(\beta-\alpha)\times\frac{1}{n+1}(x-\alpha)^{n+1}\right]_\alpha^\beta$$

$$= \frac{1}{n+2}(\beta-\alpha)^{n+2}-\frac{1}{n+1}(\beta-\alpha)^{n+2}$$

$$= \frac{(n+1)-(n+2)}{(n+2)(n+1)}(\beta-\alpha)^{n+2}$$

$$= \frac{-1}{(n+2)(n+1)}(\beta-\alpha)^{n+2}$$

$$= \boxed{-\frac{n!}{(n+2)!}(\beta-\alpha)^{n+2}}$$

(3) $\int_\alpha^\beta (x-\alpha)^n (x-\beta)^m dx = I(n, m)$ とおくと.

$$I(n,m) = \int_\alpha^\beta \left\{\frac{1}{n+1}(x-\alpha)^{n+1}\right\}' (x-\beta)^m dx$$

$$= \left[\frac{1}{n+1}(x-\alpha)^{n+1}(x-\beta)^m\right]_\alpha^\beta - \frac{1}{n+1}\int_\alpha^\beta (x-\alpha)^{n+1}\cdot m(x-\beta)^{m-1} dx$$

$$= 0 - \frac{1}{n+1}\times m\int_\alpha^\beta (x-\alpha)^{n+1}(x-\beta)^{m-1} dx$$

$$= -\frac{1}{n+1}\times m \times I(n+1, m-1)$$

$$= (-1)^2\cdot\frac{1}{n+1}\times\frac{1}{n+2}\times m(m-1)\times I(n+2, m-2)$$

$$= (-1)^{m-1}\times\frac{1}{(n+1)(n+2)\cdot\cdots\cdot(n+m-1)}\times m(m-1)(m-2)\times\cdots\cdots\times\{m-(m-1)\}\times I(n+(m-1), m-(m-1))$$

$$= (-1)^{m-1}\times\frac{n!}{(n+m-1)!}\times m!\times I(n+m-1, 1)$$ に

ここで (2) より $I(k, 1) = -\frac{k!}{(k+2)!}(\beta-\alpha)^{k+2}$ だから

$$I(n+m-1, 1) = -\frac{(n+m-1)!}{(n+m-1+2)!}(\beta-\alpha)^{n+m-1+2}.$$

より

$$I(n,m) = (-1)^{m-1}\times\frac{n!\,m!}{(n+m-1)!}\times(-1)\times\frac{(n+m-1)!}{(n+m+1)!}(\beta-\alpha)^{n+m+1}$$

$$= \boxed{(-1)^m\times\frac{n!\,m!}{(n+m+1)!}(\beta-\alpha)^{n+m+1}}$$

よって $\boxed{\text{与えられた等式はなりたつ.}}$