(1) $\pi - x = u$ とおくと $du = -dx$, $\begin{array}{c|ccc} x & 0 & \cdots & \pi \\ \hline u & \pi & \cdots & 0 \end{array}$ より

$$\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(x)\,dx$$

$$=\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(\pi-x)\,dx$$

$$=\int_{\pi}^{0}\left(\pi-u-\frac{\pi}{2}\right)f(u)\cdot(-du)$$

$$=\int_{\pi}^{0}\left(-u+\frac{\pi}{2}\right)f(u)(-du)$$

$$=\int_{\pi}^{0}\left(u-\frac{\pi}{2}\right)f(u)\cdot du$$

$$=-\int_0^{\pi}\left(u-\frac{\pi}{2}\right)f(u)\,du$$

よって $I=\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(x)dx$ とおくと

$I=-I$ がなりたつから $2I=0$ より $I=0$.

ゆえに $\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(x)\,dx=0$ がなりたつ

(2) $\displaystyle\int_0^{\pi}\frac{x\sin^3 x}{4-\cos^2 x}\,dx$

$$=\int_0^{\pi}\left(x-\frac{\pi}{2}\right)\times\frac{\sin^3 x}{4-\cos^2 x}dx+\frac{\pi}{2}\int_0^{\pi}\frac{\sin^3 x}{4-\cos^2 x}\,dx$$

ここで $f(x)=\dfrac{\sin^3 x}{4-\cos^2 x}$ とおくと $f(\pi-x)=\dfrac{\sin^3(\pi-x)}{4-\cos^2(\pi-x)}$

$$=\frac{\sin^3 x}{4-(-\cos x)^2}$$

$$=\frac{\sin^3 x}{4-\cos^2 x}=f(x)$$

で $f(x)$ は連続関数より

$$(与式)=\int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(x)\,dx+\frac{\pi}{2}\int_0^{\pi}\frac{\sin^3 x}{4-\cos^2 x}\,dx$$

$$=0+\frac{\pi}{2}\int_0^{\pi}\frac{1-\cos^2 x}{4-\cos^2 x}\times\sin x\,dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\left(\frac{4-\cos^2 x}{4-\cos^2 x}-\frac{3}{4-\cos^2 x}\right)\sin x\,dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\left(\sin x-\frac{3\sin x}{4-\cos^2 x}\right)dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\left\{\sin x-\frac{3}{4}\left(\frac{\sin x}{2+\cos x}+\frac{\sin x}{2-\cos x}\right)\right\}dx$$

$$(\text{与式}) = \frac{\pi}{2}\left\{\left[-\cos x\right]_0^{\pi} - \frac{3}{4}\left[-\log(2+\cos x)+\log(2-\cos x)\right]_0^{\pi}\right\}$$

$$= \frac{\pi}{2}\left\{-(-1)+1-\frac{3}{4}\left[\log\frac{2-\cos x}{2+\cos x}\right]_0^{\pi}\right\}$$

$$= \frac{\pi}{2}\left\{2-\frac{3}{4}\left(\log\frac{2-(-1)}{2+(-1)}-\log\frac{2-1}{2+1}\right)\right\}$$

$$= \frac{\pi}{2}\left\{2-\frac{3}{4}\left(\log 3-\log\frac{1}{3}\right)\right\}$$

$$= \frac{\pi}{2}\left(2-\frac{3}{4}\log\frac{3}{\frac{1}{3}}\right)$$

$$= \frac{\pi}{2}\left(2-\frac{3}{4}\log 3^2\right)$$

$$= \frac{\pi}{2}\left(2-\frac{3}{2}\log 3\right)$$