(1) $I_n = \int_0^1 (1-x^2)^n dx$ について

$$I_0 = \int_0^1 (1-x^2)^0 dx = \int_0^1 1\, dx = [x]_0^1 = \boxed{1}$$

$$I_1 = \int_0^1 (1-x^2)^1 dx = [x - \frac{1}{3}x^3]_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

(2) $I_n = \int_0^1 (1-x^2)^n dx$

$$= \int_0^1 (x)'(1-x^2)^n dx$$

$$= [x(1-x^2)^n]_0^1 - \int_0^1 x \times n(1-x^2)^{n-1} \cdot (-2x)\, dx$$

$$= 2n \int_0^1 x^2 (1-x^2)^{n-1} dx$$

$$= 2n \int_0^1 \{1-(1-x^2)\}(1-x^2)^{n-1} dx$$

$$= 2n \left\{ \int_0^1 (1-x^2)^{n-1} dx - \int_0^1 (1-x^2)^n dx \right\}$$

$$= 2n(I_{n-1} - I_n)$$

よって　$(1+2n)I_n = 2n\, I_{n-1}$ より

$\boxed{I_n = \frac{2n}{2n+1} I_{n-1}}$ がなりたつ。

(3) $I_3 = \frac{2\times3}{2\times3+1} I_2 = \frac{6}{7} \times \frac{2\times2}{2\times2+1} I_1 = \frac{\cancel{6}^{\,2}}{7} \times \frac{4}{5} \times \frac{2}{\cancel{3}_{\,1}} = \boxed{\frac{16}{35}}$

$I_4 = \frac{2\times4}{2\times4+1} I_3 = \frac{8}{9} \times \frac{16}{35} = \boxed{\frac{128}{315}}$

(4)
$$I_n = \frac{2n}{2\times n+1} \times \frac{2(n-1)}{2(n-1)+1} \times \cdots\cdots \times \frac{2\times 1}{2\times 1+1} \times I_0$$

$$= \frac{2\times 4\times 6\times \cdots\cdots \times 2n}{3\times 5\times 7\times \cdots\cdots \times (2n+1)} \times 1$$

$$= \frac{(2\times 4\times 6\times \cdots\cdots \times 2n)^2}{2\times 3\times 4\times 5\times 6\times \cdots\cdots \times (2n+1)}$$

$$= \frac{(2^n)^2 (n!)^2}{(2n+1)!}$$

$$= \boxed{\frac{4^n \cdot (n!)^2}{(2n+1)!}}$$