(1) $t=\tan\theta$ とおくと $dt=\dfrac{d\theta}{\cos^2\theta}$ より

$$f(1)=\int_0^1\frac{1}{1+t^2}\,dt=\int_0^{\frac{\pi}{4}}\frac{1}{1+\tan^2\theta}\times\frac{1}{\cos^2\theta}\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\cos^2\theta\cdot\frac{1}{\cos^2\theta}\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}d\theta=\boxed{\frac{\pi}{4}}$$

(2) $u=\dfrac{2t-1}{t+2}$ のとき $u=\dfrac{2(t+2)-5}{t+2}=2-\dfrac{5}{t+2}$ より

$$du=\frac{5}{(t+2)^2}\cdot dt$$

また $u=0$ のとき $2t-1=0$ より $t=\frac{1}{2}$

$u=\frac{1}{3}$ のとき $\dfrac{1}{3}=\dfrac{2t-1}{t+2}$ より $t+2=6t-3$

$5t=5$

$t=1$.

よって
$$\int_0^{\frac{1}{3}}\frac{1}{1+u^2}\,du=\int_{\frac{1}{2}}^1\frac{1}{1+\left(\frac{2t-1}{t+2}\right)^2}\cdot\frac{5}{(t+2)^2}\,dt$$

$$=\int_{\frac{1}{2}}^1\frac{5}{(t+2)^2+(2t-1)^2}\,dt$$

$$=\int_{\frac{1}{2}}^1\frac{5}{5t^2+5}\,dt$$

$$=\int_{\frac{1}{2}}^1\frac{1}{1+t^2}\,dt$$ だから

$$f\left(\tfrac{1}{2}\right)+f\left(\tfrac{1}{3}\right)=\int_0^{\frac{1}{2}}\frac{1}{1+t^2}\,dt+\int_0^{\frac{1}{3}}\frac{1}{1+u^2}\,du$$

$$=\int_0^{\frac{1}{2}}\frac{1}{1+t^2}\,dt+\int_{\frac{1}{2}}^1\frac{1}{1+t^2}\,dt$$

$$=\int_0^1\frac{1}{1+t^2}\,dt=f(1)=\boxed{\frac{\pi}{4}}$$