(1) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^2 x}\,dx$

$\displaystyle = \int_0^{\frac{\pi}{4}} \frac{\cos x}{1-\sin^2 x}\,dx$

ここで　$t = \sin x$ とおくと　$dt = \cos x \cdot dx$

$$\begin{array}{c|ccc} x & 0 & \cdots & \frac{\pi}{4} \\ \hline t & 0 & \cdots & \frac{1}{\sqrt{2}} \end{array}$$ より

$$(与式) = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-t^2}\,dt$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$$

$$= \frac{1}{2}\Big[-\log|1-t| + \log|1+t|\Big]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\left[\log\left|\frac{1+t}{1-t}\right|\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{2}\log\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right|$$

$$= \frac{1}{2}\log\left|\frac{2+\sqrt{2}}{2-\sqrt{2}}\right|$$

$$= \frac{1}{2}\log\frac{(2+\sqrt{2})^2}{2}$$

$$= \log\frac{2+\sqrt{2}}{\sqrt{2}}$$

$$= \boxed{\log(\sqrt{2}+1)}$$

(2) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^3 x} = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^4 x} dx = \int_0^{\frac{\pi}{4}} \frac{\cos x \cdot dx}{(1-\sin^2 x)^2}$

ここで $t=\sin x$ とおくと $dt=\cos x \cdot dx$, $\begin{array}{c|c} x & 0 \cdots \frac{\pi}{4} \\ \hline t & 0 \cdots \frac{1}{\sqrt{2}} \end{array}$ より

$$(与式) = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1-t^2)^2} dt$$

ここで $\displaystyle\frac{1}{(1-t^2)^2} = \frac{1}{(1+t)^2(1-t)^2} = \frac{A}{(1+t)^2} + \frac{B}{(1-t)^2} + \frac{C}{1+t} + \frac{D}{1-t}$ とおくと

$\displaystyle\frac{1}{(1-t^2)^2} = \frac{A(1-t)^2+B(1+t)^2+C(1+t)(1-t)^2+D(1+t)^2(1-t)}{(1+t)^2(1-t)^2}$ であり

分子部分だけみると

$1 = A(t^2-2t+1)+B(t^2+2t+1)+C(1+t)(1-2t+t^2)+D(1-t)(1+2t+t^2)$ より

$1 = (C-D)t^3+(A+B-C-D)t^2+(-2A+2B-C+D)t+(A+B+C+D)$

よって $\begin{cases} C-D=0 & —① \\ A+B-C-D=0 & —② \\ -2A+2B-C+D=0 & —③ \\ A+B+C+D=1 & —④ \end{cases}$ がなりたつ

④−②より $2C+2D=1$ また ①より $C=D$ だから

$2C+2C=1$ より $C=D=\frac{1}{4}$

よって ③は $-2A+2B-C+C=0$ より $A=B$ であり

④+②より $2A+2B=1$ より $2A+2A=1$ から $A=B=\frac{1}{4}$

ゆえに $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^3 x} = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{4}\left\{\frac{1}{(1+t)^2} + \frac{1}{(1-t)^2} + \frac{1}{1+t} + \frac{1}{1-t}\right\} dt$

$$= \frac{1}{4}\left[-\frac{1}{1+t} + \frac{1}{1-t} + \log|1+t| - \log|1-t|\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4}\left[\frac{2t}{1-t^2} + \log\left|\frac{1+t}{1-t}\right|\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4}\left(\frac{2\times\frac{1}{\sqrt{2}}}{1-\frac{1}{2}} + \log\left|\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right|\right)$$

$$= \frac{1}{4}\left(2\sqrt{2} + \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{1}{4}\left\{2\sqrt{2} + \log(\sqrt{2}+1)^2\right\}$$

$$= \frac{1}{4}\left\{2\sqrt{2} + 2\log(\sqrt{2}+1)\right\} = \boxed{\frac{1}{2}\left\{\sqrt{2} + \log(\sqrt{2}+1)\right\}}$$

(3) $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$

$x = \tan\theta$ とおくと　$dx = \dfrac{d\theta}{\cos^2\theta}$

x	$0 \cdots 1$
θ	$0 \cdots \frac{\pi}{4}$

よって (2)の結果をつかうと

$$(与式) = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta} \cdot \frac{d\theta}{\cos^2\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^3\theta} = \boxed{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)}$$

別解

$\boxed{\displaystyle\int \frac{1}{\cos x}\,dx}$ について

$$\cos^2 x = 1-\sin^2 x$$
$$\cos^2 x = (1+\sin x)(1-\sin x)$$

よって $\dfrac{\cos x}{1+\sin x} = \dfrac{1-\sin x}{\cos x}$ より

$$\frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

よって
$$\boxed{\int \frac{1}{\cos x}\,dx} = \int\left(\frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x}\right)dx$$
$$= \log|1+\sin x| - \log|\cos x| + C$$
$$\boxed{= \log\left|\frac{1+\sin x}{\cos x}\right| + C \quad (C\text{は定数})}$$

$\boxed{\displaystyle\int \frac{1}{\cos^3 x}\,dx}$ について

$I = \displaystyle\int \frac{dx}{\cos^3 x}$ とおくと

$$I = \int \frac{1}{\cos^2 x}\cdot\frac{1}{\cos x}\,dx$$
$$= \int (\tan x)' \times \frac{1}{\cos x}\,dx$$
$$= \tan x\cdot\frac{1}{\cos x} - \int (\tan x)\cdot\left(-\frac{-\sin x}{\cos^2 x}\right)dx$$
$$= \frac{\sin x}{\cos^2 x} - \int \frac{\sin^2 x}{\cos^3 x}\,dx$$
$$= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x}\,dx$$
$$= \frac{\sin x}{\cos^2 x} - \int \frac{dx}{\cos^3 x} + \int \frac{dx}{\cos x}$$

よって $I = \dfrac{\sin x}{\cos^2 x} - I + \log\left|\dfrac{1+\sin x}{\cos x}\right| + C'$ より

(C'は定数)

$$2I = \frac{\sin x}{\cos^2 x} + \log\left|\frac{1+\sin x}{\cos x}\right| + C'$$

ゆえに $\boxed{I = \displaystyle\int \frac{dx}{\cos^3 x} = \frac{1}{2}\times\frac{\sin x}{\cos^2 x} + \frac{1}{2}\log\left|\frac{1+\sin x}{\cos x}\right| + C \quad (C\text{は定数})}$

($\dfrac{1}{2}\log\left|\dfrac{1+\sin x}{\cos x}\right|$ の項は $\underline{\dfrac{1}{2}\log\left|\dfrac{1+\sin x}{\cos x}\right|}$ $= \dfrac{1}{4}\log\dfrac{(1+\sin x)^2}{\cos^2 x}$

$$= \frac{1}{4}\log\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}$$
$$= \underline{\frac{1}{4}\log\frac{1+\sin x}{1-\sin x}}$$ なので これでも可)