(1) $x=-t$ とおくと $dx=-dt$ で $\begin{array}{c|ccc} x & 0 & \cdots & a \\ \hline t & 0 & \cdots & -a \end{array}$ だから

$$\int_{-a}^{a}\frac{f(x)}{1+e^{-x}}dx = \int_{0}^{a}\frac{f(x)}{1+e^{-x}}dx+\int_{-a}^{0}\frac{f(x)}{1+e^{-x}}dx$$

$$=\int_{0}^{a}\frac{f(x)}{1+e^{-x}}dx+\int_{a}^{0}\frac{f(-t)}{1+e^{t}}(-dt)$$

$$=\int_{0}^{a}\frac{f(x)}{1+e^{-x}}dx-\int_{a}^{0}\frac{f(-t)}{1+e^{t}}dt$$

$$=\int_{0}^{a}\frac{f(x)}{1+e^{-x}}dx+\int_{0}^{a}\frac{f(t)}{1+e^{t}}dt \quad (\because f(t)=f(-t)より)$$

$$=\int_{0}^{a}\frac{e^{x}f(x)}{e^{x}(1+e^{-x})}dx+\int_{0}^{a}\frac{f(x)}{1+e^{x}}dx$$

$$=\int_{0}^{a}\frac{e^{x}f(x)}{1+e^{x}}dx+\int_{0}^{a}\frac{f(x)}{1+e^{x}}dx$$

$$=\int_{0}^{a}\frac{(1+e^{x})f(x)}{1+e^{x}}dx$$

$$=\int_{0}^{a}f(x)dx$$

よって $\displaystyle\int_{-a}^{a}\frac{f(x)}{1+e^{-x}}dx=\int_{0}^{a}f(x)dx$ はなりたつ。

(2) (1)を利用すると

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{x\sin x}{1+e^{-x}}dx = \int_{0}^{\frac{\pi}{2}}x\sin x\,dx$$

$$=\int_{0}^{\frac{\pi}{2}}x(-\cos x)'dx$$

$$=\left[-x\cos x\right]_{0}^{\frac{\pi}{2}}+\int_{0}^{\frac{\pi}{2}}\cos x\,dx$$

$$=0+\left[\sin x\right]_{0}^{\frac{\pi}{2}}$$

$$=\boxed{1}$$