

$$(1) \quad f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \dots + \log(1+x^{2^n}) + \dots$$

この無限級数の第 n 項までの和を S_n とすると

$$\begin{aligned} S_n &= \log(1+x) + \log(1+x^2) + \log(1+x^4) + \dots + \log(1+x^{2^n}) \\ &= \log \{ (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n}) \} \\ &= \log \{ 1+x+x^2+x^3+x^4+\dots+x^{1+2+2^2+\dots+2^n} \} \end{aligned}$$

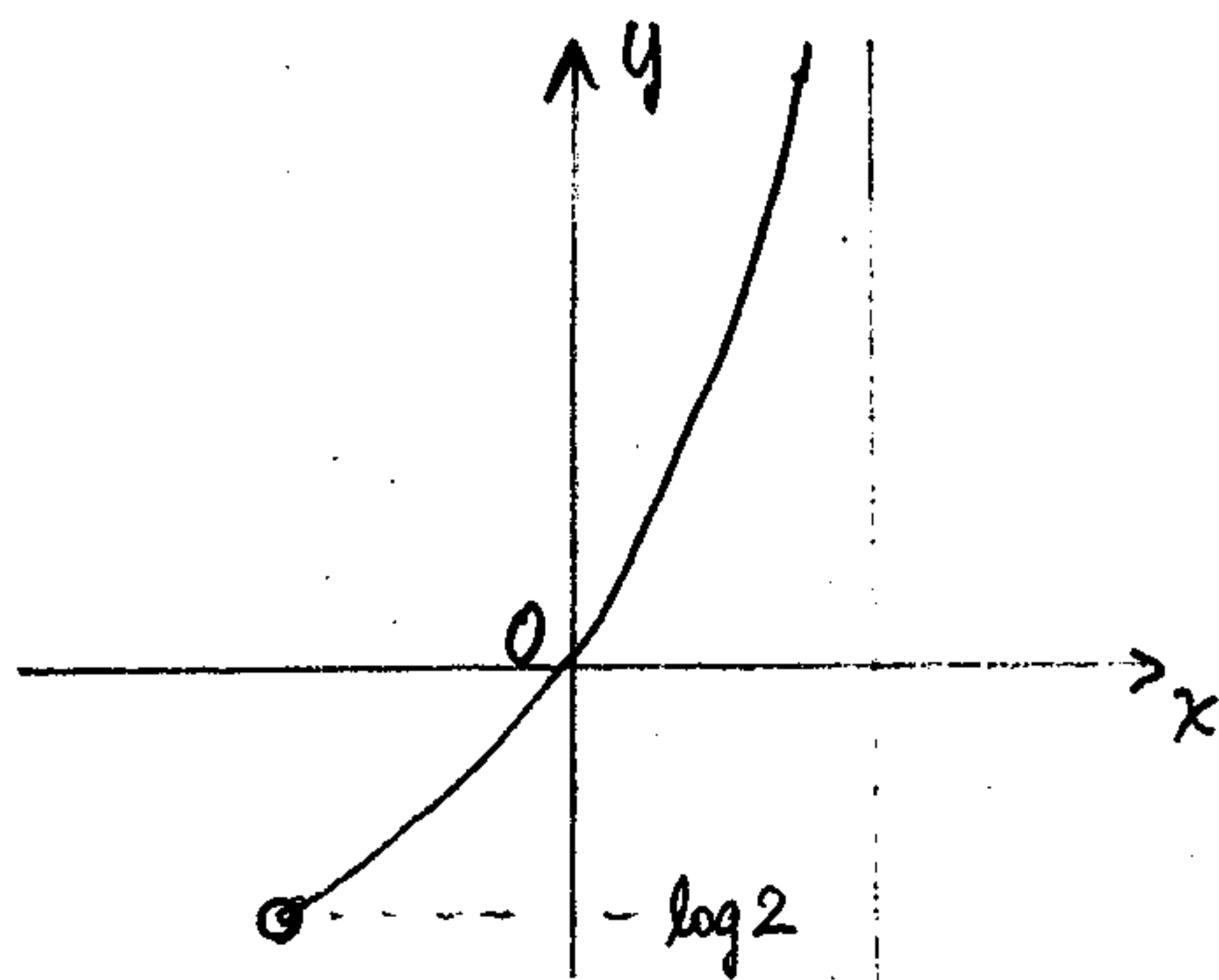
$1+x+x^2+\dots+x^{1+2+2^2+\dots+2^n}$ が収束すれば"よい"
これは公比 x の級数だから

$$\text{収束条件は } \boxed{|x| < 1}$$

$$\begin{aligned} (2) \quad f(x) &= \lim_{n \rightarrow \infty} S_n \\ &= \lim_{n \rightarrow \infty} \log(1+x+x^2+x^3+x^4+\dots+x^{2^{n+1}}-1) \\ &= \lim_{n \rightarrow \infty} \log\left(\frac{1-x^{2^{n+1}}}{1-x}\right) \end{aligned}$$

$$|x| < 1 \text{ なら } \lim_{n \rightarrow \infty} x^{2^{n+1}} = 0 \text{ だから}$$

$$\begin{aligned} f(x) &= \log \frac{1}{1-x} \\ &= \log(1-x)^{-1} \\ &= \boxed{-\log(1-x)} \end{aligned}$$



$$\begin{aligned} (3) \quad &\int f(x) dx \\ &= - \int \log(1-x) dx \\ &= - \int \{ -(1-x) \}' \log(1-x) dx \end{aligned}$$

$$= (1-x) \log(1-x) - \int (1-x) \times \frac{-1}{1-x} dx$$

$$= (1-x) \log(1-x) + \int dx$$

$$= \boxed{(1-x) \log(1-x) + x + C}$$