

$$\frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2} \quad \text{とおく}$$

$$\frac{1}{x(x-1)^2} = \frac{a(x-1)^2 + b x(x-1) + c x}{x(x-1)^2}$$

分子部分は

$$1 = a(x-1)^2 + b x(x-1) + c x$$

$$x=0 \text{ 代入して}$$

$$1 = a$$

$$x=1 \text{ 代入して}$$

$$1 = c$$

$$x=2 \text{ 代入して}$$

$$1 = a + 2b + 2c$$

$$\text{よって } 1 = 1 + 2b + 2$$

$$2b = -2$$

$$b = -1$$

$$\text{ゆえに } \begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases} \quad \text{だから}$$

$$\begin{aligned} \int \frac{dx}{x(x-1)^2} &= \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right) dx \\ &= \log|x| - \log|x-1| - \frac{1}{x-1} + C \end{aligned}$$

$$= \boxed{\log \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + C} \quad (C \text{ は積分定数})$$