$C_n = (n+1)\int_0^1 x^n \cos\pi x\,dx \quad (n=1,2,\cdots)$ において

(1) $C_{n+2} = (n+3)\int_0^1 x^{n+2}\cos\pi x\,dx$

$= (n+3)\int_0^1 x^{n+2}\left(\frac{1}{\pi}\sin\pi x\right)'dx$

$= (n+3)\left\{\left[x^{n+2}\times\frac{1}{\pi}\sin\pi x\right]_0^1 - \int_0^1 (n+2)x^{n+1}\left(\frac{1}{\pi}\sin\pi x\right)dx\right\}$

$= -\frac{1}{\pi}(n+3)(n+2)\int_0^1 x^{n+1}\sin\pi x\,dx$

$= -\frac{1}{\pi}(n+3)(n+2)\int_0^1 x^{n+1}\left(-\frac{1}{\pi}\cos\pi x\right)'dx$

$= -\frac{1}{\pi}(n+3)(n+2)\left\{\left[x^{n+1}\times\left(-\frac{1}{\pi}\right)\cos\pi x\right]_0^1 - \int_0^1 (n+1)x^n\times\left(-\frac{1}{\pi}\right)\cos\pi x\,dx\right\}$

$= -\frac{1}{\pi}(n+3)(n+2)\left\{1\times\left(-\frac{1}{\pi}\right)\times(-1) + \frac{1}{\pi}\times(n+1)\int_0^1 x^n\cos\pi x\,dx\right\}$

$= -\frac{1}{\pi}(n+3)(n+2)\left(\frac{1}{\pi}+\frac{1}{\pi}C_n\right)$

よって $\boxed{C_{n+2} = -\frac{1}{\pi^2}(n+3)(n+2)(1+C_n)}$ である

(2) (1)より $C_n+1 = \frac{-\pi^2}{(n+2)(n+3)}C_{n+2}$ であるので

$0 \leqq |C_n+1| = \left|\frac{\pi^2}{(n+2)(n+3)}C_{n+2}\right|$ である

また $|C_{n+2}| = (n+1)\left|\int_0^1 x^n\cos\pi x\,dx\right| \leqq (n+1)\int_0^1 x^n\times 1\,dx$

$= (n+1)\times\frac{1}{n+1}\left[x^{n+1}\right]_0^1 = 1.$

よって $0 \leqq |C_n+1| \leqq \left|\frac{\pi^2}{(n+2)(n+3)}\right|$ がなりたつ.

$\lim_{n\to\infty}\frac{\pi^2}{(n+2)(n+3)} = 0$ より $\lim_{n\to\infty}|C_n+1| = 0$

ゆえに $\boxed{\lim_{n\to\infty} C_n = -1}$ となる

(3) $C = \lim_{n\to\infty} C_n = -1$ より $\lim_{n\to\infty}\frac{C_{n+1}-C}{C_n-C} = \lim_{n\to\infty}\frac{C_{n+1}+1}{C_n+1}$

$= \lim_{n\to\infty}\frac{\frac{-\pi^2}{(n+3)(n+4)}C_{n+3}}{\frac{-\pi^2}{(n+2)(n+3)}C_{n+2}} = \lim_{n\to\infty}\frac{n+2}{n+4}\times\frac{C_{n+3}}{C_{n+2}}$

$= \lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{4}{n}}\times\frac{C_{n+3}}{C_{n+2}} = \frac{1+0}{1+0}\times\frac{-1}{-1} = \boxed{1}$

である